

A Faster Algorithm for the Maximum Even Factor Problem

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Abstract. Given a digraph $G = (VG, AG)$, an *even factor* $M \subseteq AG$ is a subset of arcs that decomposes into a collection of node-disjoint paths and even cycles. Even factors in digraphs were introduced by Geelen and Cunningham and generalize path matchings in undirected graphs.

Finding an even factor of maximum cardinality in a general digraph is known to be NP-hard but for the class of *odd-cycle symmetric* digraphs the problem is polynomially solvable. So far, the only combinatorial algorithm known for this task is due to Pap; it has the running time of $O(n^4)$ (hereinafter n stands for the number of nodes in G).

In this paper we present a novel *sparse recovery* technique and devise an $O(n^3 \log n)$ -time algorithm for finding a maximum cardinality even factor in an odd-cycle symmetric digraph.

1 Introduction

In [4] Cunningham and Geelen introduced the notion of *independent path matchings* and investigated their connection to the separation algorithms for the matchable set polytope, which was previously studied by Balas and Pulleyblank [1]. Finding an independent path matching of maximum size was recognized as an intriguing example of a graph-theoretic optimization problem that is difficult to tackle by the purely combinatorial means. Two algorithms were given by Cunningham and Geelen: one relies on the ellipsoid method [4], and the other is based on deterministic evaluations of the Tutte matrix [5]. Later, a rather complicated combinatorial algorithm was proposed by Spille and Weismantel [13].

The notion of an *even factor* was introduced as a further generalization of path matchings in a manuscript of Cunningham and Geelen [6] (see also [3]). An even factor is a set of arcs that decomposes into a node-disjoint collection of simple paths and simple cycles of even lengths. Since cycles of length 2 are allowed, it is not difficult to see that finding a maximum matching in an undirected graph G reduces to computing a maximum even factor in the digraph obtained from G by replacing each edge with a pair of oppositely directed arcs. On the other hand, no reduction from even factors to non-bipartite matchings is known.

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Finding a maximum cardinality even factor is known to be NP-hard in general digraphs [3]. For the class of *weakly symmetric digraphs* a min-max relation was established by Cunningham and Geelen [6] and by Pap and Szegő [12]. Later it was noted by Pap [10] that these arguments hold for a slightly broader class of *odd-cycle symmetric digraphs*. Takazawa and Kobayashi [16] pointed out a relation between even factors and jump systems and showed that the requirement for a digraph to be odd-cycle symmetric is natural, in a sense.

The question of finding a combinatorial solution to the maximum even factor problem in an odd-cycle symmetric digraph stood open for quite a while until Pap gave a direct $O(n^4)$ -time algorithm [10,11]. His method can be slightly sped up to $O(n^2(m + n \log n))$, as explained in Section 3. (Hereinafter n stands for the number of nodes in G and m denotes the number of arcs.) To compare: the classical algorithm of Micali and Vazirani for finding a maximum non-bipartite matching, which is a special case of the maximum even factor problem, runs in $O(mn^{1/2})$ time [9]. It is tempting to design a faster algorithm for the maximum even factor problem by applying the ideas developed for matchings (e.g. blocking augmentation [7]). There are, however, certain complications making even the bound of $O(mn)$ nontrivial.

To explain the nature of these difficulties let us briefly review Pap's approach (a more detailed exposition will be given in Section 3). It resembles Edmonds' non-bipartite matching algorithm and executes a series of iterations each trying to increase the size of the current even factor M by one. At each such iteration, a search for an augmenting path P is performed. If no such path is found then M is maximum. Otherwise, the algorithm tries to apply P to M . If no odd cycle appears after the augmentation then the iteration completes. Otherwise, a certain contracting reduction is applied to G and M .

Hence, each iteration consists of *phases* and the number of nodes in the current digraph decreases with each phase. Totally there are $O(n)$ iterations and $O(n)$ phases during each iteration, which is quite similar to the usual blossom-shrinking method. The difference is that during a phase the reduction may completely change the alternating reachability structure so the next phase is forced to start looking for P from scratch. (Compare this with Edmonds' algorithm where a blossom contraction changes the alternating forest in a predictable and consistent way thus allowing this forest to be reused over the phases.)

In this paper we present a novel $O(n^3 \log n)$ -time algorithm for solving the maximum even factor problem. It is based on Pap's method but grows the alternating forest in a more careful fashion. When a contraction is made in the current digraph the forest gets destroyed. However, we are able to restore it by running a *sparse recovery* procedure that carries out a reachability search in a specially crafted digraph with $O(n)$ arcs in $O(n \log n)$ time (where the $\log n$ factor comes from manipulations with data structures).

Our method seems applicable to a large variety of related problems. In particular, the $O(mn^3)$ -time algorithm of Takazawa [14] solves the weighted even factor problem in $O(mn^3)$ time and also involves recomputing the alternating forest from scratch on each phase.

Similar effects of reachability failure are known to occur in the maximum C_4 -free 2-factor problem [11]. Also, adding matroidal structures leads to the *maximum independent even factor problem*, which is solvable by the methods similar to the discussed above, see [8]. All these problems can benefit from the sparse recovery technique. Due to the lack of space we omit the details on these extensions.

2 Preliminaries

We employ some standard graph-theoretic notation throughout the paper. For an undirected graph G , we denote its sets of nodes and edges by VG and EG , respectively. For a directed graph, we speak of arcs rather than edges and denote the arc set of G by AG . A similar notation is used for paths, trees, and etc. We allow parallel edges and arcs but not loops. As long as this leads to no confusion, an arc from u to v is denoted by (u, v) .

A path or a cycle is called *even* (respectively *odd*) if it consists of an even (respectively *odd*) number of arcs or edges. For a digraph G a *path-cycle matching* is a subset of arcs M that is a union of node-disjoint simple paths and cycles in G . When M contains no odd cycle it is called an *even factor*. The *size* of M is its cardinality and the *maximum even factor problem* prompts for constructing an even factor of maximum size.

An arc (u, v) in a digraph G is called *symmetric* if (v, u) is also present in G . Following the terminology from [10], we call G *odd-cycle symmetric* (respectively *weakly symmetric*) if for each odd (respectively any) cycle C all the arcs of C are symmetric. As already noted in Section 1, the maximum even factor problem is NP-hard in general but is tractable for odd-cycle symmetric digraphs.

Maximizing the size of an even factor M in a digraph G is equivalent to minimizing its *deficiency* $\text{def}(G, M) := |VG| - |M|$. The minimum deficiency of an even factor in G is called the *deficiency* of G and is denoted by $\text{def}(G)$.

For a digraph G and $U \subseteq VG$, the set of arcs entering (respectively leaving) U is denoted by $\delta_G^{\text{in}}(U)$ and $\delta_G^{\text{out}}(U)$. Also, we write $\gamma_G(U)$ to denote the set of arcs with both endpoints in U and $G[U]$ to denote the subgraph of G induced by U , i.e. $G[U] = (U, \gamma_G(U))$. When, the digraph is clear from the context it is omitted from notation.

To *contract* a set $U \subseteq VG$ in a digraph G means to replace nodes in U by a single *complex node*. The arcs in $\gamma(VG - U)$ are not affected, arcs in $\gamma(U)$ are dropped, and the arcs in $\delta^{\text{in}}(U)$ (respectively $\delta^{\text{out}}(U)$) are redirected so as to enter (respectively leave) the complex node. The resulting graph is denoted by G/U . We identify the arcs in G/U with their pre-images in G . Note that G/U may contain multiple parallel arcs but not loops. If G' is obtained from G by an arbitrary series of contractions then $G' = G/U_1/\dots/U_k$ for a certain family of disjoint subsets $U_1, \dots, U_k \subseteq VG$ (called the *maximum contracted sets*).

Algorithm 1 SIMPLE-AUGMENT(G, M)

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1: Search for an augmenting path  $P$  in  $\vec{G}(M)$ 
2: if  $P$  does not exist then
3:   return NULL
4: else if  $P$  exists and is feasible then
5:   return  $M \triangle A(P)$ 
6: else  $\{P$  exists but is not feasible $\}$ 
7:   Put  $M_i \leftarrow M \triangle A(P_i)$  for  $i = 0, \dots, k + 1$ 
8:   Find an index  $i$  such that  $M_i$  is an even factor while  $M_{i+1}$  is not
9:   Find the unique odd cycle  $C$  in  $M_{i+1}$ 
10:   $G' \leftarrow G/C, M' \leftarrow M_i/C$ 
11:   $\overline{M}' \leftarrow \text{SIMPLE-AUGMENT}(G', M')$ 
12:  if  $\overline{M}' = \text{NULL}$  then  $\{M'$  is maximum in  $G'\}$ 
13:    return NULL
14:  else  $\{M'$  is augmented in  $G'$  to a larger even factor  $\overline{M}'\}$ 
15:    Undo the contractions and transform  $\overline{M}'$  to an even factor  $M^+$  in  $G$ 
16:    return  $M^+$ 
17:  end if
18: end if
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3 Pap's Algorithm

Consider an odd-cycle symmetric digraph G . The algorithm for finding a maximum even factor in G follows the standard scheme of cardinality augmentation. Namely, we initially start with the empty even factor M and execute a series of *iterations* each aiming to increase $|M|$ by one. Iterations call SIMPLE-AUGMENT routine that, given an odd-cycle symmetric digraph G and an even factor M in G either returns a larger even factor M^+ or NULL indicating that the maximum size is reached.

3.1 Augmentations

Let us temporarily allow odd cycles and focus on path-cycle matchings in G . The latter are easily characterized as follows. Construct two disjoint copies of VG : $V^1 := \{v^1 \mid v \in VG\}$ and $V^2 := \{v^2 \mid v \in VG\}$. For each arc $a = (u, v) \in AG$ add the edge $\{u_1, v_2\}$ (*corresponding* to a). Denote the resulting undirected bipartite graph by \tilde{G} .

Clearly, a path-cycle matching M in G is characterized by the following properties: for each node $v \in VG$, M has at most one arc entering v and also at most one arc leaving v . Translating this to \tilde{G} one readily sees that M generates a matching \tilde{M} in \tilde{G} . Moreover, this correspondence between matchings in \tilde{G} and path-cycle matchings in G is one-to-one. A node u^1 (respectively u^2) in \tilde{G} not covered by \tilde{M} is called a *source* (a *sink*, respectively).

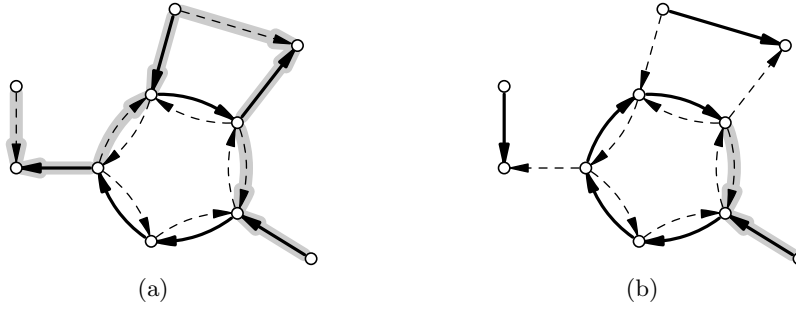


Fig. 1. Preparing for a contraction. Subfigure (a): the arcs of M are bold and the grayed arcs correspond to path P_{i+1} . Subfigure (b): path P_i is applied, the arcs of M_i are bold, and the grayed arcs indicate the remaining part of P_{i+1} .

Given a digraph G and a path-cycle matching M in G we turn \tilde{G} into a digraph $\vec{G}(M)$ by directing the edges $\{u^1, v^2\}$ corresponding to arcs $(u, v) \in M$ from v^2 to u^1 and the other edges from u^1 to v^2 .

Definition 1 A simple path in $\vec{G}(M)$ starting in a source node is called alternating. An alternating path ending in a sink node is called augmenting.

For an alternating path P let $A(P)$ denote the set of arcs in G corresponding to the arcs of P in $\vec{G}(M)$. Hereinafter $A \triangle B$ denotes the symmetric difference of sets A and B . The next statements are well-known.

Claim 1 If $\vec{G}(M)$ admits no augmenting path then M is a path-cycle matching of maximum size.

Claim 2 If P is an augmenting (respectively an even alternating) path in $\vec{G}(M)$ then $M' := M \triangle A(P)$ is path-cycle matching obeying $|M'| = |M| + 1$ (respectively $|M'| = |M|$).

The augmentation procedure (see Algorithm 1) constructs $\vec{G}(M)$ and searches for an augmenting path P there. In case no such path exists, the current even factor M is maximum by Claim 1 (even in the broader class of path-cycle matchings), hence the algorithm terminates. Next, consider the case when $\vec{G}(M)$ contains an augmenting path P . Claim 2 indicates how a larger path-cycle matching M' can be formed from M , however M' may contain an odd cycle. The next definition focuses on this issue.

Definition 2 Let P be an augmenting or an even alternating path. Then P is called feasible if $M' := M \triangle A(P)$ is again an even factor.

If P is feasible then SIMPLE-AUGMENT exits with the updated even factor $M \triangle A(P)$. Consider the contrary, i.e. P is not feasible. Clearly, P is odd, say it

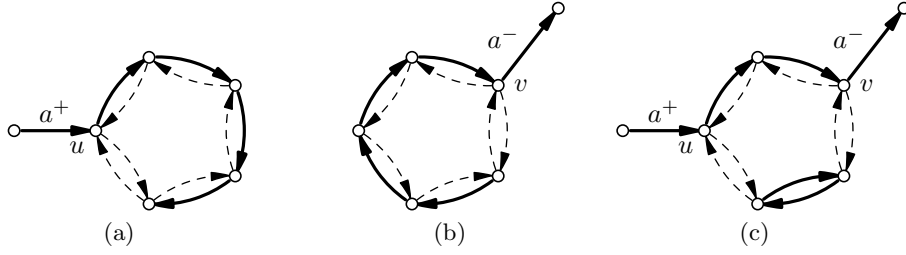


Fig. 2. Some cases appearing in Lemma 1. Digraph G and even factor N (bold arcs) are depicted.

consists of $2k + 1$ arcs. Construct a series of even alternating paths P_0, \dots, P_k where P_i is formed by taking the first $2i$ arcs of P ($0 \leq i \leq k$). Also, put $P_{k+1} := P$ and $M_i := M \triangle A(P_i)$ ($0 \leq i \leq k + 1$).

Then there exists an index i ($0 \leq i \leq k$) such that P_i is feasible while P_{i+1} is not feasible. In other words, M_i is an even factor obeying $\text{def}(G, M_i) = \text{def}(G, M)$ and M_{i+1} contains an odd cycle. Since M_i and M_{i+1} differ by at most two arcs, it can be easily showed that an odd cycle in M_{i+1} , call it C , is unique (see [10]). Moreover, M_i fits C , that is, $|M_i \cap AC| = |VC| - 1$ and $\delta^{\text{out}}(VC) \cap M_i = \emptyset$. See Fig. 1 for an example.

It turns out that when an even factor fits an odd cycle then a certain optimality-preserving contraction is possible. As long as no confusion is possible, for a digraph H and a cycle K we abbreviate H/VK to H/K . Also, for $X \subseteq AH$ we write X/K to denote $X \setminus \gamma_H(VK)$.

Claim 3 (Pap [10]) *Let K be an odd cycle in H and N be an even factor that fits K . Put $H' := H/K$ and $N' := N/K$. Then H' is an odd-cycle symmetric digraph and N' is an even factor in H' . Moreover, if N' is maximum in H' then N is maximum in H .*

Note that M is maximum in G if and only if M_i is maximum in G . SIMPLE-AUGMENT contracts C in G . Let $G' := G/C$ and $M' := M_i/C$ denote the resulting digraph and the even factor. To check if M' is maximum in G' a recursive call SIMPLE-AUGMENT(G', M') is made. If NULL is returned then M' is a maximum even factor in G' , which implies by Claim 3 that the initial even factor M was maximum in G . In this case SIMPLE-AUGMENT terminates returning NULL.

Suppose that the recursive call was able to augment M' to a larger even factor \overline{M}' in G' . The algorithm transforms \overline{M}' to an even factor in G with the help of the following statement from [10] (see Fig. 2 for examples):

Lemma 1. *Let K be an odd cycle in an odd-cycle symmetric digraph H . Put $H' := H/K$ and let N' be an even factor in H' . Then there exists an even factor N in H obeying $\text{def}(H, N) = \text{def}(H', N')$.*

3.2 Complexity

There are $O(n)$ iterations each consisting of $O(n)$ phases. To bound the complexity of a single phase note that it takes $O(m)$ time to find an augmenting path P (if the latter exists). We may construct all the path-cycle matchings M_0, \dots, M_{k+1} and decompose each of them into node-disjoint paths and cycles in $O(n^2)$ time. Hence, finding the index i and the cycle C takes $O(n^2)$ time. Contracting C in G takes $O(m)$ time. (We have already spent $O(m)$ time looking for P , so it is feasible to spend another $O(m)$ time to construct the new digraph $G' = G/C$ explicitly.) An obvious bookkeeping allows to undo all the contractions performed during the iteration and transform the final even factor in the contracted digraph an even factor the initial digraph in $O(m)$ time. Totally, the algorithm runs in $O(n^4)$ time.

The above bound can be slightly improved as follows. Note that the algorithm needs an arbitrary index i such that M_i is an even factor and M_{i+1} is not, i.e. i is not required to be minimum. Hence, we may carry out a binary search over the range $[0, k + 1]$. At each step we keep a pair of indices (l, r) such that M_l is an even factor while M_r is not. Replacing the current segment $[l, r]$ by a twice smaller one takes $O(n)$ time and requires constructing and testing a single path-cycle matching M_t where $t := \lfloor (l + r)/2 \rfloor$. This way, the $O(n^2)$ term reduces to $O(n \log n)$ and the total running time of becomes $O(n^2(m + n \log n))$. The ultimate goal of this paper is to get rid of the $O(m)$ term.

4 A Faster Algorithm

4.1 Augmentations

The bottleneck of SIMPLE-AUGMENT are the augmenting path computations. To obtain an improvement we need better understanding of how these paths are calculated. Similarly to the usual path-finding algorithms we maintain a directed out-forest \mathcal{F} rooted at the source nodes. The nodes belonging to this forest are called \mathcal{F} -reachable. At each step a new arc (u, v) leaving an \mathcal{F} -reachable node u is scanned and either gets added to \mathcal{F} (thus making v \mathcal{F} -reachable) or skipped because v is already \mathcal{F} -reachable. This process continues until a sink node is reached or no unscanned arcs remain in the digraph.

Definition 3 *Let G be a digraph and M be an even factor in G . An alternating forest \mathcal{F} for M is a directed out-forest in $\vec{G}(M)$ such that: (i) the roots of \mathcal{F} are all the source nodes in $\vec{G}(M)$; (ii) every path from a root of \mathcal{F} to a leaf of \mathcal{F} is even.*

The intuition behind the suggested improvement is to grow \mathcal{F} carefully and avoid exploring infeasible alternating paths.

Definition 4 *An alternating forest \mathcal{F} is called feasible if every even alternating or augmenting path in \mathcal{F} is feasible. An alternating forest \mathcal{F} is called complete if it contains no sink node and for each arc (u, v) in $\vec{G}(M)$ if u is \mathcal{F} -reachable then so is v .*

We replace SIMPLE-AUGMENT by a more sophisticated recursive FAST-AUGMENT procedure. It takes an odd-cycle symmetric digraph G , an even factor M in G , and an additional flag named *sparsify*. The procedure returns a digraph \overline{G} obtained from G by a number of contractions and an even factor \overline{M} in \overline{G} . Additionally, it may return an alternating forest $\overline{\mathcal{F}}$ for \overline{M} in \overline{G} . Exactly one of the following two cases applies:

- (1) $\text{def}(\overline{G}, \overline{M}) = \text{def}(G, M) - 1$ and $\overline{\mathcal{F}}$ is undefined;
- (2) $\text{def}(\overline{G}, \overline{M}) = \text{def}(G, M)$, \overline{M} is maximum in \overline{G} , M is maximum in G , and $\overline{\mathcal{F}}$ is a proper complete alternating forest for \overline{M} in \overline{G} .

Assuming the above properties are true, let us explain how FAST-AUGMENT can be used to perform a single augmenting iteration. Given a current even factor M in G the algorithm calls FAST-AUGMENT(G, M, TRUE) and examines the result. If $\text{def}(\overline{G}, \overline{M}) = \text{def}(G, M)$ then by (2) M is a maximum even factor in G , the algorithm stops. (Note that the forest $\overline{\mathcal{F}}$, which is also returned by FAST-AUGMENT, is not used here. This forest is needed due to the recursive nature of FAST-AUGMENT.) Otherwise $\text{def}(\overline{G}, \overline{M}) = \text{def}(G, M) - 1$ by (1); this case will be referred to as a *breakthrough*. Applying Lemma 1, \overline{M} is transformed to an even factor M^+ in G such that $\text{def}(G, M^+) = \text{def}(\overline{G}, \overline{M}) = \text{def}(G, M) - 1$. This completes the current iteration.

Clearly, the algorithm constructs a maximum even factor correctly provided that FAST-AUGMENT obeys the contract. Let us focus on the latter procedure. It starts growing a feasible alternating forest \mathcal{F} rooted at the source nodes (line 1). During the course of the execution, FAST-AUGMENT *scans* the arcs of G in a certain order. For each node u in G we keep the list $L(u)$ of all unscanned arcs leaving u . The following invariant is maintained:

- (3) if $a = (u, v)$ is a scanned arc then either $a \in M$ or both u^1 and v^2 are \mathcal{F} -reachable.

Consider an \mathcal{F} -reachable node u^1 . To enumerate the arcs leaving u^1 in $\overrightarrow{G}(M)$ we fetch an unscanned arc $a = (u, v)$ from $L(u)$. If $a \in M$ or v^2 is \mathcal{F} -reachable then a is skipped and another arc is fetched. (In the former case a does not generate an arc leaving u^1 in $\overrightarrow{G}(M)$, in the latter case v^2 is already \mathcal{F} -reachable so a can be made scanned according to (3).)

Otherwise, consider the arc $a_1 := (u^1, v^2)$ in $\overrightarrow{G}(M)$ and let P_0 denote the even alternating feasible path from a root of \mathcal{F} to u^1 . Note that each node x^2 in $\overrightarrow{G}(M)$ (for $x \in VG$) is either a sink or has the unique arc leaving it. A *single* step occurs when v^2 is a sink (lines 10–13). The algorithm constructs an augmenting path $P_1 = P_0 \circ a_1$ leading to v^2 . (Here $L_1 \circ L_2$ stands for the concatenation of L_1 and L_2 .) If P_1 is feasible, the current even factor gets augmented according to Claim 2 and FAST-AUGMENT terminates. Otherwise, forest growing stops and the algorithm proceeds to line 22 to deal with a contraction.

A *double* step is executed when v^2 is not a sink (lines 15–20). To keep the leafs of \mathcal{F} on even distances from roots, \mathcal{F} it must be extended by adding pairs

Algorithm 2 FAST-AUGMENT($G, M, sparsify$)

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1: Initialize forest  $\mathcal{F}$ 
2: for all unscanned arcs  $a = (u, v)$  such that  $u^1 \in V\mathcal{F}$  do
3:   Mark  $a$  as scanned
4:   if  $a \in M$  or  $v^2 \in V\mathcal{F}$  then
5:     continue for {to line 2}
6:   end if
7:    $a_1 \leftarrow (u^1, v^2)$ 
8:   Let  $P_0$  be the even alternating path to  $u^1$  in  $\mathcal{F}$  { $P_0$  is feasible}
9:   if  $v^2$  is a sink then {single step}
10:     $P_1 \leftarrow P_0 \circ a_1$  { $P_1$  is augmenting}
11:    if  $P_1$  is feasible then
12:      return  $(G, M \triangle A(P_1), \text{NULL})$ 
13:    end if
14:  else {double step}
15:    Let  $a_2 = (v^2, w^1)$  be the unique arc leaving  $v^2$  { $w^1 \notin V\mathcal{F}$ }
16:     $P_1 \leftarrow P_0 \circ a_1 \circ a_2$  { $P_1$  is even alternating}
17:    if  $P_1$  is feasible then
18:      Add nodes  $v^2$  and  $w^1$  and arcs  $a_1, a_2$  to  $\mathcal{F}$ 
19:      continue for {to line 2}
20:    end if
21:  end if
22:   $M_0 \leftarrow M \triangle A(P_0), M_1 \leftarrow M \triangle A(P_1)$ 
23:  Find a unique odd cycle  $C$  in  $M_1$ 
24:   $G' \leftarrow G/C, M' \leftarrow M_0/C$ 
25:  if  $sparsify = \text{FALSE}$  then
26:    return FAST-AUGMENT( $G', M', \text{FALSE}$ )
27:  end if
28:  Construct the digraph  $H'$ 
29:   $(\overline{H}', \overline{M}', \overline{\mathcal{F}}) \leftarrow \text{FAST-AUGMENT}(H', M', \text{FALSE})$ 
30:  Compare  $V\overline{H}'$  and  $VG$ : let  $Z_1, \dots, Z_k$  be the maximum contracted sets and
     $z_1, \dots, z_k$  be the corresponding complex nodes in  $\overline{H}'$ 
31:   $\overline{G} \leftarrow G/Z_1/\dots/Z_k$ 
32:  if  $\text{def}(\overline{G}, \overline{M}') < \text{def}(G', M')$  then
33:    return  $(\overline{G}, \overline{M}', \text{NULL})$ 
34:  end if
35:  Unscan the arcs in  $\overline{G}'$  that belong to  $M$  and the arcs that enter  $z_1, \dots, z_k$ 
36:   $G \leftarrow \overline{G}', M \leftarrow \overline{M}', \mathcal{F} \leftarrow \overline{\mathcal{F}}$ 
37: end for
38: return  $(G, M, \mathcal{F})$ 
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of arcs. Namely, there is a unique arc leaving v^2 in $\overrightarrow{G}(M)$, say $a_2 = (v^2, w^1)$ (evidently $(w, v) \in M$). Moreover, w^1 is not a source node and (v^2, w^1) is the only arc entering w^1 . Hence, w^1 is not \mathcal{F} -reachable. If $P_1 := P_0 \circ a_1 \circ a_2$ is feasible then a_1 and a_2 are added to \mathcal{F} thus making v^2 and w^1 \mathcal{F} -reachable. Otherwise, a contraction is necessary.

Now we explain how the algorithm deals with contractions at line 22. One has an even alternating feasible path P_0 and an infeasible augmenting or even alternating path P_1 (obtained by extending P_0 by one or two arcs). Put $M_0 := M \triangle A(P_0)$ and $M_1 := M \triangle A(P_1)$. Let C denote the unique odd cycle in M_1 . Put $G' := G/C$, $M' := M_0/C$. If *sparsify* = FALSE then FAST-AUGMENT acts similar to SIMPLE-AUGMENT, namely, it makes a recursive call passing G' and M' as an input and, hence, restarting the whole path computation.

Next, suppose *sparsify* = TRUE. In this case the algorithm tries to *recover* some proper alternating forest for the contracted digraph G' and the updated even factor M' . To accomplish this, a sparse digraph H' is constructed and FAST-AUGMENT is recursively called for it (with *sparsify* = FALSE). The latter nested call may obtain a breakthrough, that is, find an even factor of smaller deficiency. In this case, the outer call terminates immediately. Otherwise, the nested call returns a complete proper alternating forest $\overline{\mathcal{F}}$ for an even factor \overline{M}' in a digraph \overline{H}' (obtained from H' by contractions). This forest is used by the outer call to continue the path-searching process. It turns out that almost all of the arcs that were earlier fetched by the outer call need no additional processing and may remain scanned w.r.t. the new, recovered forest. This way, the algorithm amortizes arc scans during the outer call.

More formally, the algorithm constructs H' as follows. First, take the node set of G , add all the arcs of M and all the arcs $(u, v) \in AG$ such that (u^1, v^2) is present in \mathcal{F} . Denote the resulting digraph by H . We need to ensure that H is odd-cycle symmetric: if some arc (u, v) is already added to H and the reverse arc (v, u) exists in G then add (v, u) to H . Next, put $H' := H/C$. Note that H' is a sparse spanning subgraph of G' (i.e. $VH' = VG'$, $|AH'| = O(n)$) and M' is an even factor in H' .

The algorithm makes a recursive call FAST-AUGMENT(H', M', FALSE). Let \overline{H}' and \overline{M}' be the resulting digraph and the even factor, respectively. Compare the node sets of \overline{H}' and H . Clearly, \overline{H}' is obtained from H by a number of contractions (G/C being one of them). Let Z_1, \dots, Z_k be the maximum disjoint subsets of G such that $\overline{H}' = G/Z_1/\dots/Z_k$. Also, let z_1, \dots, z_k be the composite nodes in \overline{H}' corresponding to Z_1, \dots, Z_k . The algorithm applies these contractions to G and constructs the digraph $\overline{G}' := G/Z_1/\dots/Z_k$. Clearly \overline{M}' is an even factor in both \overline{G}' and \overline{H}' . If $\text{def}(\overline{H}', \overline{M}') < \text{def}(H', M') = \text{def}(G, M)$, then one has a breakthrough, FAST-AUGMENT terminates yielding \overline{G}' and \overline{M}' .

Otherwise, the recursive call in line 29 also returns a proper complete forest $\overline{\mathcal{F}}$ for \overline{H}' and \overline{M}' . Recall that some arcs in G are marked as *scanned*. Since we identify the arcs of \overline{G}' with their pre-images in G , one may speak of scanned arcs of \overline{G}' . The algorithm “unscans” certain arcs $a = (u, v) \in A\overline{G}'$ by adding them back to their corresponding lists $L(u)$ to ensure (3). Namely, the arcs that belong to M and are present in \overline{G}' and the arcs that enter any of the complex nodes z_1, \dots, z_k in \overline{G}' are unscanned. After this, the algorithm puts $G := \overline{G}'$, $M := \overline{M}'$, $\mathcal{F} := \overline{\mathcal{F}}$ and proceeds with growing \mathcal{F} (using the adjusted set of scanned arcs).

Finally, if FAST-AUGMENT has scanned all the arcs of G and is unable to reach a sink, the resulting forest \mathcal{F} is both proper and complete. In particular, by (3) no augmenting path for M exists. By Claim 3 this implies the maximality of M in G . The algorithm returns the current digraph G , the current (maximum) even factor M , and also the forest \mathcal{F} , which certifies the maximality of M .

The correctness of FAST-AUGMENT is evident except for the case when it tries to recover \mathcal{F} and alters the set of scanned arcs. One has to prove that (3) holds for the updated forest and the updated set of the scanned arcs. The proof of this statement is given in Section A.

4.2 Complexity

We employ arc lists to represent digraphs. When a subset U in a digraph Γ is contracted we enumerate all arcs incident to U and update the lists accordingly. If a pair of parallel arcs appears after contraction, these arcs are merged, so all our digraphs remain simple. The above contraction of U takes $O(|V\Gamma| \cdot |U|)$ time. During FAST-AUGMENT the sum of sizes of the contracted subsets telescopes to $O(n)$, so graph contractions take $O(n^2)$ time in total. The usual bookkeeping allows to undo contractions and recover a maximum even factor in the original digraph in $O(m)$ time.

Consider an invocation FAST-AUGMENT(Γ, N, FALSE) and let us bound its complexity (including the recursive calls). The outer loop of the algorithm (lines 2–37) enumerates the unscanned arcs. Since *sparsify* = FALSE, each arc can be scanned at most once, so the bound of $O(|A\Gamma|)$ for the number of arc scans follows. Using the appropriate data structures to represent even factors the reachability checks in lines 11 and 17 can be carried out in $O(\log |V\Gamma|)$ time (see Section B for more details). Constructing M_0 , M_1 , and C takes $O(|V\Gamma|)$ time. This way, FAST-AUGMENT(Γ, N, FALSE) takes $O((k+1) |A\Gamma| \log |V\Gamma|)$ time, where k denotes the number of graph contractions performed during the invocation.

Next, we focus on FAST-AUGMENT(Γ, N, TRUE) call. Now one may need to perform more than $|A\Gamma|$ arc scans since forest recovery may produce new unscanned arcs (line 35). Note that forest recovery totally occurs $O(|V\Gamma|)$ times (since each such occurrence leads to a contraction). During each recovery M generates $O(|V\Gamma|)$ unscanned arcs or, in total, $O(|V\Gamma|^2)$ such arcs for the duration of FAST-AUGMENT. Also, each node z_i generates $O(|V\Gamma|)$ unscanned arcs (recall that we merge parallel arcs and keep the current digraph simple). The total number of these nodes processed during FAST-AUGMENT is $O(|V\Gamma|)$ (since each such node corresponds to a contraction). Totally these nodes produce $O(|V\Gamma|^2)$ unscanned arcs. Hence, the total number of arc scans is $O(|A\Gamma| + |V\Gamma|^2) = O(|V\Gamma|^2)$.

Each feasibility check costs $O(\log |V\Gamma|)$ time, or $O(|V\Gamma|^2 \log |V\Gamma|)$ in total. Finally, we must account for the time spent in the recursive invocations during FAST-AUGMENT(Γ, N, TRUE). Each such invocation deals with a sparse digraph and hence takes $O((k+1) |V\Gamma| \log |V\Gamma|)$ time (where, as earlier, k denotes the number of contractions performed by the recursive invocation). Since the to-

tal number of contractions is $O(|VT|)$, the sum over all recursive invocations telescopes to $O(|VT|^2 \log |VT|)$.

The total time bound for FAST-AUGMENT(Γ, N, TRUE) (including the recursive calls) is also $O(|VT|^2 \log |VT|)$. Therefore a maximum even factor in an odd-cycle symmetric digraph can be found in $O(n^3 \log n)$ time, as claimed.

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Appendix

A Correctness

In order to establish the correctness of the algorithm one needs to prove that once the forest \mathcal{F} gets recovered at line 36 of FAST-AUGMENT the set of scanned arc obeys property (3). The latter is equivalent to the following:

Lemma 2. *Consider line 36 in Algorithm 2 and let $a = (u, v)$ be an arbitrary arc of \overline{G}' . Then either a is unscanned or $a \in \overline{M}'$ or both u^1 and v^2 are $\overline{\mathcal{F}}$ -reachable.*

First, we shall need a more convenient characterization of reachable nodes. Consider a digraph Γ and a node $v \in V\Gamma$. Construct a new odd-cycle symmetric digraph $\Gamma * v$ from Γ by adding a new node v' and an arc (v, v') .

Lemma 3. *For each Γ and v either $\text{def}(\Gamma * v) = \text{def}(\Gamma)$ or $\text{def}(\Gamma * v) = \text{def}(\Gamma) + 1$.*

Proof.

Each even factor N in Γ is also an even factor in $\Gamma * v$, hence $\text{def}(\Gamma * v) \leq \text{def}(\Gamma) + 1$. Also, for an even factor N^* in $\Gamma * v$ put $N := N^* \setminus \{(v, v')\}$. Then, N is an even factor in Γ obeying $|N| \geq |N^*| - 1$. This implies $\text{def}(\Gamma) \leq \text{def}(\Gamma * v)$, as required. \square

Lemma 4. *Let N be a maximum even factor in Γ and \mathcal{T} be a feasible alternating forest for N . If v^1 is \mathcal{T} -reachable then $\text{def}(\Gamma * v) = \text{def}(\Gamma)$.*

Proof.

Consider the even alternating path P from a root of \mathcal{T} to v^1 . Since \mathcal{T} is feasible, $N' := N \triangle A(P)$ is an even factor in Γ and no arc of N' leaves v' . Now $N^* := N' \cup \{(v, v')\}$ is an even factor in $\Gamma * v$. Therefore, $\text{def}(\Gamma * v) \leq \text{def}(\Gamma * v, N^*) = \text{def}(\Gamma, N) = \text{def}(\Gamma)$ implying $\text{def}(\Gamma * v) = \text{def}(\Gamma)$ by Lemma 3. \square

Lemma 5. *Let N be a maximum even factor in Γ and \mathcal{T} be a complete alternating forest for N . If v^1 is not \mathcal{T} -reachable then $\text{def}(\Gamma * v) = \text{def}(\Gamma) + 1$ and N is a maximum even factor in $\Gamma * v$.*

Proof.

The alternating forest \mathcal{T}^* obtained from \mathcal{T} by adding a new root node $(v')^1$ is complete for $\Gamma * v$ and N . Hence, N is maximum in both Γ and $\Gamma * v$ and $\text{def}(\Gamma * v) = \text{def}(\Gamma)$. \square

Consider a node $v \in VG$. If $v \in VG - (Z_1 \cup \dots \cup Z_k)$ then we say that v *survives the contractions*. These nodes are both present in G and \overline{G}' .

Lemma 6. *Suppose that a node $v \in VG$ survives the contractions and v^1 is \mathcal{F} -reachable. Then v^1 is $\overline{\mathcal{F}}$ -reachable. Also, the nodes z_1^1, \dots, z_k^1 are $\overline{\mathcal{F}}$ -reachable.*

Proof.

The transformation of H to \overline{H}' and of M to \overline{M}' may be viewed as follows:

$$(4) \quad \begin{array}{ccccccc} (H, M) & (H, M_0) & (H', M') & & (\overline{H}', \overline{M}') \\ \parallel & \parallel & \parallel & & \parallel \\ (\Gamma^0, N^0) & \rightarrow (\Gamma^0, N_0^0) & \rightarrow (\Gamma^1, N^1) & \rightarrow (\Gamma^1, N_0^1) & \rightarrow \dots & \rightarrow (\Gamma^s, N^s) \end{array}$$

Here $\Gamma^0, \dots, \Gamma^s$ are odd-cycle symmetric digraphs, N^i, N_0^i are even factors in Γ^i obeying $|N^i| = |N_0^i|$. Each N_0^i fits some odd cycle C^i in Γ^i and $\Gamma^{i+1} = \Gamma^i / C^i$, $N^{i+1} = N_0^i / C^i$.

Recall that the nested call to FAST-AUGMENT in line 29 did not result into a breakthrough, so \overline{M}' is maximum in \overline{H}' and $\overline{\mathcal{F}}$ is a complete alternating forest.

Let us prove the first claim of the lemma. Consider a node v surviving the contractions such that v^1 is \mathcal{F} -reachable in G . Suppose towards a contradiction that v^1 is not $\overline{\mathcal{F}}$ -reachable in \overline{H}' . By Lemma 5, $\overline{M}' = N^s$ is a maximum even factor in $\overline{H}' * v = \Gamma^s * v$. Then, by Claim 3, N_0^{s-1} is a maximum even factor in $\Gamma^{s-1} * v$ and hence so is N^{s-1} (by the equality of sizes). Proceeding this way in the backward direction we conclude that N^i is a maximum even factor in $\Gamma^i * v$ for all $i = 0, \dots, s$. In particular, $N^0 = M$ is maximum in $\Gamma^0 = H$. Since $(v, v') \notin M$, M is also maximum in H and $\text{def}(H * v, M) = \text{def}(H, M) + 1$. This contradicts Lemma 4 and the fact that v^1 is \mathcal{F} -reachable.

For the second claim, fix a node $v = z_i$ and suppose that v^1 is not $\overline{\mathcal{F}}$ -reachable in \overline{H}' . Consider the sequence of transformations (4) and suppose that v was created as a complex node in Γ^j while contracting an odd cycle C^{j-1} in Γ^{j-1} . As indicated above, N^j is a maximum even factor in $\Gamma^j * v$. The latter, however, is false since N_0^{j-1} fits C^{j-1} and, therefore, N^j has no arcs leaving v (cf. Fig. 1(b) for an example). Hence, N^j can be enlarged to $N^j \cup \{(v, v')\}$ — a contradiction. \square

Lemma 7. *Suppose that a node $v \in VG$ survives the contractions and v^2 is \mathcal{F} -reachable. Then v^2 is $\overline{\mathcal{F}}$ -reachable.*

Proof.

The node v^2 cannot be a source, hence it is reached by some arc (u^1, v^2) in \mathcal{F} , where u^1 is \mathcal{F} -reachable and $a = (u, v) \notin M$. Let $a_0 = (u_0, v)$ be the image of a under contractions. Note that $a \in AH$ and $a_0 \in A\overline{H}'$. By Lemma 6, u_0^1 is $\overline{\mathcal{F}}$ -reachable. Therefore, if $a_0 \notin \overline{M}'$ then the completeness of $\overline{\mathcal{F}}$ implies that v^2 is $\overline{\mathcal{F}}$ -reachable. It remains to consider the case $a_0 \in \overline{M}'$. The node u_0^1 is not a source (since $a_0 \in \overline{M}'$ leaves u_0) but is $\overline{\mathcal{F}}$ -reachable. In the auxiliary bipartite digraph u_0^1 is entered by exactly one arc, namely (v^2, u_0^1) . Hence, v^2 must be $\overline{\mathcal{F}}$ -reachable, as required. \square

Finally, we present the desired correctness proof.

Proof of Lemma 2.

Consider an arc $a = (u, v) \in \overline{AG}'$. If a is not scanned then we are done. Otherwise let $a_0 = (u_0, v_0)$ denote its pre-image in G . Here $u = u_0$ if u_0 survives the contractions or $u = z_i$ if $u_0 \in Z_i$. Also, since all arcs entering the complex nodes z_1, \dots, z_k are unscanned (line 35), v_0 survives the contractions and hence $v = v_0$. Since the algorithm only decreases the set of scanned arcs, a_0 must also be scanned in G . Clearly $a_0 \notin M$ since all the arcs that belong to M and are present in \overline{G}' were as unscanned. Therefore, both u_0 and $v_0 = v$ are \mathcal{F} -reachable in G by the invariant (3). Applying Lemma 6 to u_0 and Lemma 7 to v_0 we see that both u and v must be $\overline{\mathcal{F}}$ -reachable. The proof is now complete. \square

B Feasibility Checks

This section explains how path feasibility checks, which are performed by FAST-AUGMENT at lines 11 and 17, can be made efficient. That is, given an even factor N and a feasible even alternating path P_0 we need to verify that an even alternating or an augmenting path P_1 (obtained from P_0 by appending one or two arcs) is feasible. We make use of a certain data structure \mathcal{D} that maintains an even factor $M \triangle A(P_0)$ as a collection of node-disjoint paths and cycles. The following operations are supported by \mathcal{D} :

- INSERT(u, v): assuming that u is the end node of some path P_u in \mathcal{D} and v is the start node of some path P_v in \mathcal{D} , add the arc (u, v) thus linking P_u and P_v or, in case $P_u = P_v$, turning this path into a cycle;
- REMOVE(u, v): assuming that $a = (u, v)$ is an arc belonging to some path or cycle in \mathcal{D} , remove a thus splitting the path into two parts or turning the cycle into a path.
- IS-ODD-CYCLE(u, v): assuming that $a = (u, v)$ is an arc belonging to some path or cycle in \mathcal{D} , check if a belongs to an odd cycle.

We make use of balanced search trees additionally augmented with SPLIT and CONCATENATE operations (e.g. red-black trees, see [2,15]) to represent paths and cycles in \mathcal{D} . This way, INSERT, REMOVE, and IS-ODD-CYCLE take $O(\log |V\Gamma|)$ time each. Now checking if P_1 for feasibility is done by calling INSERT(u, v) and, in case of a double step, REMOVE(w, v), and finally making IS-ODD-CYCLE(u, v) request. If the latter indicates that P_1 is not feasible, the changes in \mathcal{D} are rolled back.

During a FAST-AUGMENT(Γ, N, FALSE) call we grow \mathcal{F} in a depth-first fashion and maintain the structure \mathcal{D} corresponding to the current \mathcal{F} -reachable node u^1 (i.e. \mathcal{D} keeps the decomposition of $N \triangle A(P)$ where P is the path in \mathcal{F} from a root to u^1). When \mathcal{F} gets extended by arcs (u^1, v^2) and (v^2, w^1) , w^1 becomes the new current node and \mathcal{D} is updated accordingly by the above INSERT(u, v) and REMOVE(w, v) calls. When the algorithm backtracks from w^1 to u^1 , changes in \mathcal{D} are reverted. This way, each feasibility check costs $O(\log |V\Gamma|)$ time

Next, consider a $\text{FAST-AUGMENT}(\Gamma, N, \text{TRUE})$ call. The above time bound of $O(\log |V\Gamma|)$ per check is only valid if we grow \mathcal{F} from scratch. However, the algorithm also reuses the forest that is returned by the nested FAST-AUGMENT call in line 29. This incurs an overhead of $O(|V\Gamma| \log |V\Gamma|)$ per forest recovery (this additional time is needed to traverse the arcs that are present in the recovered forest \mathcal{F} and update \mathcal{D} accordingly). There are $O(|V\Gamma|)$ forest recoveries during the call so the total overhead is $O(|V\Gamma|^2 \log |V\Gamma|)$ time. This does not affect the time bound of $\text{FAST-AUGMENT}(\Gamma, N, \text{TRUE})$.